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DUALITY IN CONTACT PROBLEMS

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The paper is concerned with the dual formulation of the contact problems dealt with earlier in [1, 2]. The Young — Fenchel — Moreau transformation is used in the perturbations of the initial formulations.

1. Basic definitions and theorems [3]. We consider the following problem (problem P): to find

$$\inf_{v \in V} J(v), \quad J: V \to R$$

where V is a Banach (below it is Hilbert) space and R is a set of real numbers. We note that problem P contains the formulations of the problems of mutual contact of elastic bodies obtained in [1, 2], provided that we set

$$J(v) = \begin{cases} J(v), & v \in K \\ +\infty, & v \overline{\in} K \end{cases}$$

where K is a subset of V (definition and the properties of K are given in [1, 2].

The dual formulations are constructed as follows [3]. We take a space V^* dual to V with respect to the bilinear form $\langle , \rangle_V$ and a pair of spaces Y and Y^* placed in duality by the bilinear form $\langle , \rangle_Y$. We denote the elements of Y and Y^* by p and p^*. We construct a functional $\Phi(v, p)$, $\Phi: V \times Y \to R$, such that $\Phi(v, 0) = J(v)$, and consider the problem (problem P_p) of finding

$$\inf_{v \in V} \Phi(v, p)$$

We call the latter problem a perturbation of the problem P (and the functional Φ a perturbation of J).

Let us now apply the Young transformation to the functional Φ

$$\Phi^*(v^*, p^*) = \sup_{v \in V,\, p \in Y} \{\langle v^*, v \rangle_V + \langle p^*, p \rangle_Y - \Phi(v, p)\} \qquad (1.1)$$

Problem P^* consists of finding

$$\sup_{p^* \in Y^*} \{-\Phi^*(0, p^*)\}$$

and is a dual of P.

Theorem 1. Let V be a reflexive Banach space, $\Phi \not\equiv +\infty$, and let $u_0 \in V$ exist such that $p \to \Phi(u_0, p)$ is finite and continuous at zero ($p \in Y$), $\lim J(v) = +\infty$ when $\|v\| \to +\infty$. Then every problem P and P^* has at least one solution and

$$\inf_{v\in V} J(v) = \sup_{p^*\in Y^*} \{-\Phi^*(0, p^*)\}$$
$$\Phi(\bar{u}, 0) + \Phi^*(0, \bar{p}^*) = 0$$

where $\bar{u}$ is a solution of P and $\bar{p}^*$ is a solution of P^*.
The functional

$$L^\circ(v, p^*) = -\sup_{p\in Y} \{\langle p^*, p\rangle_Y - \Phi(v, p)\} \tag{1.2}$$

will be called the Lagrangian L° of problem P relative to the given perturbation Φ.

Theorem 2. When Theorem 1 holds, the following assertions are equivalent:
a) $\bar{u}$ and $\bar{p}^*$ and solutions of the problems P and P^*,
b) the pair $(\bar{u}, \bar{p}^*)$ is a saddle point of the Lagrangian L° on $V \times Y$, i.e.

$$L^\circ(\bar{u}, p^*) \leqslant L^\circ(\bar{u}, \bar{p}^*) \leqslant L^\circ(v, \bar{p}^*), \ \forall v, p^*$$

2. The problem of contact between a linearly elastic body and a perfectly rigid stamp. Here we have [1]

$$V = \{v \mid v(x) \in H^1(\Omega);\ v(x) = 0,\ x \in S_u\} \tag{2.1}$$
$$K = \{v \mid \Psi(x) + v(x)\cdot\nabla\Psi(x) \geqslant 0,\ \forall x \in S_c\}$$
$$J(v) = {}^1/_2\, a(v, v) - L(v), \quad a(v, v) =$$
$$\int_\Omega a_{ijkh}\varepsilon_{kh}(v)\,\varepsilon_{ij}(v)\,d\Omega, \quad L(v) = \int_\Omega \rho F v\, d\Omega + \int_{S_\sigma} P v\, dS$$

We use the results of [2] to rewrite the expression for K as follows:

$$K = \{v \mid \delta(x) - v_N(x) \geqslant 0,\ \forall\, x \in S_c\}$$
$$\delta(x) = \Psi(x) / |\nabla\Psi(x)|,\ v_N = v\cdot\nabla\Psi / |\nabla\Psi|$$

Below we assume that $\rho F \in L_2(\Omega)$, $P \in L_2(S_\sigma)$ and, that the boundary S of the region Ω is regular (the regularity is defined in ch. 5 of [4]). The hypothesis

$$a_{ijkh}\varepsilon_{kh}\varepsilon_{ij} \geqslant c\ \varepsilon_{ij}\varepsilon_{ij}, \quad c = \text{const} > 0$$

together with the assumption that mes $S_u > 0$ ensures that the solution exists and is unique (the problem is analyzed for $S_u = \varnothing$ in [2]).

We shall consider here three types of perturbations leading to new problems amenable to numerical solution.

1) Arrow–Hurwicz - type perturbation

$$\Phi_1(v, p) = J(v) + \chi_\varepsilon(v, p) \tag{2.2}$$

where χ_ε is the indicatrix function of the set

$$\varepsilon = \{(v, p) \in V \times Y \mid p \in Y,\ v_N(x) - p(x) \leqslant \delta(x),\ x \in S_c\}$$
$$Y = \{p \mid p = p(x), x \in S_c, p \in L_2(S_c)\}$$

The form $\langle , \rangle_Y$ is represented here by a scalar product in $L_2(S_c)$.

Carrying out the computations in accordance with the formulas of Sect. 1 (using a scalar product in $L_2(\Omega)$ as $\langle , \rangle_V$), we arrive at the following dual problem P_1^*: to find

$$\sup_{p^* \leqslant 0} \inf_{v \in V} [J(v) + \int_{S_c} (\delta - v_N) p^* dS]$$

We note that the "old" variables could not be eliminated here. As a result, we arrive at the problem of finding the saddle point, and use of Theorem 2 leads to the same problem. We also note that the conjugate (dual) variable p^* describes in the present case the distribution of contact pressure.

2) Castigliano-type perturbation. Here we assume that

$$\Phi_2(v, p) = \frac{1}{2} \int_\Omega a_{ijkh} [\varepsilon_{kh}(v) - p_{kh}] [\varepsilon_{ij}(v) - p_{ij}] d\Omega - L(v) \qquad (2.3)$$

provided that $v \in K$ and $\Phi_2(v, p) = +\infty$ when $v \notin K$. Therefore p is a tensor quantity and

$$Y = \{p \mid p_{ij} = p_{ji}, p_{ij} = p_{ij}(x), x \in \Omega\}$$

Since Y^* is defined here with respect to the bilinear form

$$\int_\Omega p_{ij}^* p_{ij} d\Omega$$

it follows that the conjugate variables are, in this case, the components of the stress tensor.

Carrying out the computations with help of the formulas of Sect. 1, we arrive at the following dual problem P_2^*: to find

$$\sup_{\sigma \in M} \left\{ -\frac{1}{2} \int_\Omega A_{ijkh} \sigma_{kh} \sigma_{ij} d\Omega + \int_{S_c} \sigma_N \delta dS \right\}$$

$$M = \{\sigma \mid \operatorname{div} \sigma + \rho F = 0; \ \sigma \cdot \nu|_{S_\sigma} = P; \ \sigma_N|_{S_c} \leqslant 0, \ \sigma_T|_{S_c} = 0\}$$

The dual variables follow the accepted notation: σ is the stress tensor and σ_{ij} denote its components in the Cartesian coordinate system, A_{ijkh} is the tensor of compliance moduli, i.e. $\varepsilon_{ij} = A_{ijkh} \sigma_{kh}$, $\sigma_N = (\sigma \cdot \nu) \cdot \nu$, $\sigma_T = (\sigma \cdot \nu) - \sigma_N \nu$, and ν is normal to S.

3) Let us now combine the perturbations of the type 1) and 2) in the following manner:

$$\Phi_3(v, p) = \Phi_2(v, p_1) + \chi_\varepsilon(v, p_2) \qquad (2.4)$$

assuming also that in this case Φ_2 is defined by (2.3) everywhere and not only on K (the requirement that $v \in K$ is introduced by means of the function χ_ε). Let us assume that

$$p = \{p_1, p_2\} \in Y_1 \times Y_2 \qquad (2.5)$$

$$\langle p^*, p \rangle = \langle p_1^*, p_1 \rangle + \langle p_2^*, p_2 \rangle \qquad (2.6)$$

Carrying out the necessary computations and replacing, as in case 2), p_1^* by σ and p_2^* by σ_N, we obtain the following dual proposition (problem P_3^*): to find

$$\sup_{\sigma\in M}\inf_{v\in V}\left\{-\frac{1}{2}\int_\Omega A_{ijkh}\sigma_{kh}\sigma_{ij}d\Omega+\int_{S_c}(\delta-v_N)\,\sigma_N dS\right\}$$

N o t e 1. The problems P_2^* and P_3^* are difficult in the sense that when they are solved approximately (using e.g. the finite elements method), then the equilibrium equations must be satisfied inside the region Ω . For this reason, the change to a formulation using the Lagrangians seems sensible. Using the definition (2.6) we find, that

$$L_2^\circ(v,\ p^*)=-\int_\Omega\left[\frac{1}{2}A_{ijkh}\sigma_{kh}\sigma_{ij}-\sigma_{ij}\varepsilon_{ij}(v)\right]d\Omega-L(v) \tag{2.7}$$

$$L_3^\circ(v,\ p^*)=L_2^{\circ\circ}(v,\ p^*)+\int_{S_c}(-\delta+v_N)\,\sigma_N dS \tag{2.8}$$

($L_2^{\circ\circ}$ given by the formula (2.7) everywhere and not only for $v\in K$; if $v\,\overline{\in}\,K$, then $L_2^\circ=+\infty$; formulating the problem with L_1° obtained from (1.2) for $\Phi=\Phi_1$, does not yield anything new).

Using Theorem 2, we obtain the following problems of determining the saddle point of the Lagrangian:

$$P_{iL}^*:\ \sup_{p^*\in Y^*}\inf_{v\in V}L_i^\circ(v,\ p^*)$$
$$P_{iL}:\ \inf_{v\in V}\sup_{p^*\in Y^*}L_i^\circ(v,\ p^*),\quad i=2,3$$

N o t e 2. If the condition that $v(x)=0$ on S_u is replaced by the condition $u(x)=g(x)$, then an additional term of the type

$$\int_{S_u} g\cdot\sigma\cdot\nu dS$$

appears in the expression for Φ^* is certain obvious cases.

3. **G e n e r a l i z a t i o n t o t h e c a s e o f t h e d e f o r m a t i o n t h e o r y o f p l a s t i c i t y w i t h o u t u n l o a d i n g.** The problem of a body in contact with a perfectly rigid, smooth immovable stamp, has the following corresponding functional:

$$J_g(v)=J(v)-j(v) \tag{3.1}$$

$$j(v)=3\mu\int_0^{e_u(v)}\left[\int_\Omega\omega(s)\,sd\Omega\right]ds \tag{3.2}$$

where $J(v)$ is given by the formula (2.1). The perturbation 1) of Sect. 2 leads to the problem P_1^* where $J(v)$ is replaced by $J_g(v)$; when a Castigliano-type perturbation is used, the following additional term appears in the expression for $\Phi_2^*(0,\ p^*)$:

$$\frac{2}{3E}\int\limits_{0}^{\sigma_u(\sigma)}\left[\int\limits_{\Omega}\omega^\circ(\zeta)\,\zeta d\Omega\right]d\zeta \tag{3.3}$$

$$\omega^\circ = \omega^\circ(\sigma_u) = 3E\varphi^{-1}(\sigma_u)\,/\,(2\sigma_u) - (1+n)$$

where E is the Young's modulus, $\sigma_u(\sigma)$ is the stress intensity, n is the Poisson's ratio and the operator φ^{-1} determines the dependence of the deformation intensity on the stress intensity. An additional term also appears in the expressions for $\Phi_3^*(0, p^*)$ and the Lagrangians L_2° and L_3°.

The results established in [1] enable us to conclude that Theorems 1 and 2 can be applied to the problems discussed in Sects. 2 and 3, with unique results.

4. Problem of several elastic bodies in mutual contact. It was established in [2] that in this case we must minimise the functional

$$J(v) = \sum_\alpha J^\alpha(v) \tag{4.1}$$

under the restrictions

$$v_N^\alpha + v_N^\beta \leqslant \delta \tag{4.2}$$

where $J^\alpha(v^\alpha)$ are functionals of the type (2.1) or (3.1), α and β denote the numbers of the bodies in contact and δ is the initial gap between the bodies Ω^α and Ω^β (more accurate definitions are given in [2]).

Perturbations of the type 1) – 3) yield the following corresponding dual problems:

$$P_1^*:\ \sup_{p_\alpha^*\leqslant 0}\ \inf_{v\in V}\left[J(v) + \sum_\alpha \int\limits_{S_c^\alpha}(\delta - v_N^\alpha - v_N^\beta)p_\alpha^* dS\right]$$

$$P_2^*:\ \sup_{\sigma^\alpha\in M^\alpha}\left\{\sum_\alpha\left[-\frac{1}{2}\int\limits_{\Omega^\alpha}A_{ijkh}\sigma_{kh}\sigma_{ij}d\Omega + \int\limits_{S_c^\alpha}\sigma_N\delta dS\right]\right\}$$

$$P_3^*:\ \sup_{\sigma^\alpha\in M^\alpha}\ \inf_{v\in V}\left\{\sum_\alpha\left[-\frac{1}{2}\int\limits_{\Omega^\alpha}A_{ijkh}\sigma_{kh}\sigma_{ij}d\Omega + \int\limits_{S_c^\alpha}(\delta - v_N^\alpha - v_N^\beta)\,\sigma_N dS\right]\right\}$$

Expressions for the Lagrangians and the corresponding formulations can also be easily obtained from the formulas (2.7) and (2.8) by summation over all bodies of the system. Here again the results of [2] enable us to apply Theorems 1 and 2 and the latter yield, in particular, the proof of existence and uniqueness of the solution.

We note that the problem discussed here contains, as a particular case, a problem important in practice, of contact between an elastic body and a perfectly rigid stamp in the case when the stamp is mobile and the principal vector and moment of the forces acting in the stamp are both given. The stamp displacement field is determined by two constant vectors, the vectors of rigid translation and rotation, and in the dual formulations these vectors appear in the integrals over S_c.

5. Note on the contact problem in the theory of perfect Hencky plasticity. In the theory in question the defining equation represents a relation connecting the total deformation tensor ε_{ij} and the plastic deformation tensor λ_{ij}, and the stress tensor σ_{ij} given by (5.1), where the stress field must fulfil the condition of plasticity (5.2) and the plastic deformation satisfies the inequality (5.3) [5]:

$$\varepsilon_{ij} = A_{ijkh}\sigma_{kh} + \lambda_{ij} \tag{5.1}$$

$$F(\sigma_{ij}) \leqslant 0 \tag{5.2}$$

$$\lambda_{ij}(\tau_{ij} - \sigma_{ij}) \leqslant 0, \quad \forall\, \tau_{ij} = \tau_{ji}, \quad F(\tau_{ij}) \leqslant 0 \tag{5.3}$$

As we know [5], the problem of finding the stress field under a restricted plastic flow reduces to minimization of the Castigliano functional under the constraint (5.2), therefore in the contact problems, in absence of the body flow *in toto*, we arrive at a P_3^*-type problem under the additional constraint (5.2). The latter assertion can be established directly. Indeed, let us integrate the inequality (5.3) over the region Ω and replace the quantity λ_{ij} by $\varepsilon_{ij} - A_{ijkh}\sigma_{kh}$. Using the Green's formula, the equations of equilibrium and the boundary conditions for the quantities σ and τ, we find, that

$$\int_\Omega A_{ijkh}\sigma_{kh}(\tau_{ij} - \sigma_{ij})\, d\Omega \geqslant \int_{S_c} (\tau_{ij} - \sigma_{ij})\, u_i \nu_j\, dS \tag{5.4}$$

We assume for simplicity, that $u|_{S_u} = 0$, $\operatorname{mes} S_u > 0$.

Let us write the condition of impermeability in the form $u_N \leqslant \delta$ and note that

$$(\tau_N - \sigma_N)\, u_N \geqslant (\tau_N - \sigma_N)\, \delta \tag{5.5}$$

(we have the strict equality at the points of contact, and $(\tau_N - \sigma_N)\, u_N \geqslant \tau_N \delta)$ on the free surface). Using the representation $(\tau_{ij} - \sigma_{ij})\, u_i \nu_j = (\tau_T - \sigma_T)\, u_T + (\tau_N - \sigma_N)\, u_N$, the condition of absence of friction and the inequality (5.5) we conclude, that the problem in question is equivalent to P_3^* under the additional constraint (5.2).

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